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Dimension of homogeneous rational self-similar measures with overlaps ☆

Víctor Ruiz

Departamento de Estadística e I.O. III, Escuela Universitaria de Estadística, Universidad Complutense de Madrid, Avda. Pta. de Hierro s/n, 28040 Madrid, Spain

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ABSTRACT

We develop a tool to analyse some self-similar measures with overlaps, those obtained from systems of homotheties with centres in a lattice where the contraction ratios are all equal to the inverse of a natural number L. We obtain the local dimension of the measure as the Shannon entropy of an associated hidden Markov chain divided by the logarithm of L. This result turns out to be useful in the study of the absolute continuity or singularity of the measure, and provides two sequences converging to the dimension of the measure, one of them non-increasing and the other non-decreasing, which allows us to obtain estimates of the dimension.

1. Introduction

Self-similar measures, and the wider class of invariant measures for iterated function systems, are geometric projections of Bernoulli processes. They have played a key role in the development of fractal geometry for two main reasons: they serve as geometric models for some natural phenomena and they provide a simple mathematical framework for the development of fractal geometry concepts and methods (global and local dimension, overlaps, coverings, packings, etc.).

The case of self-similar measures with the open set condition (without overlaps; see Section 2) is the simplest and is widely understood, as it permits the construction of an isomorphism of measure spaces with the Bernoulli process underlying the code space (see [17, Remark 2.5]). Its properties are obtained from properties of Bernoulli processes, such as the strong law of large numbers. Nevertheless, the open set condition imposes restrictive geometric conditions that limit its practical applications.

The challenge is to understand more about the case of overlapping measures, for which the mathematical study presents many difficulties.

Examples of this type of measures are those associated to Bernoulli convolutions, which are self-similar measures constructed from homogeneous systems of two homotheties on the real line. The large number of articles treating Bernoulli convolutions (see [21]), starting with the work of Jessen and Wintner [13], gives an idea of the complexity of the still developing study of this problem.

Other recently studied cases of self-similar measures with overlapping and centres in a lattice are outlined below, all for the real line.

Ngai [18] considers the homogenous systems of homotheties with ratio $1/2$, related to the dilation equation in wavelet theory. Lau and Ngai [15] consider some homogeneous systems of homotheties with ratio $1/L$ where $L \geqslant 3$ is an integer. For examples of related research, see [24,26]. In all these works, multifractal properties of the associated self-similar measures are studied.

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E-mail address: vruiz@estad.ucm.es.

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We deal here with the general case of homogeneous systems of homotheties in $\mathbb{R}^d$ with centres in a lattice and ratio $1/L$ where $L \geqslant 2$ is integer, which includes the cases studied in [15,18]. We study the dimension of the self-similar measure μ.

We will find a way of analysing μ by relating it with an associated ergodic hidden Markov chain. To do so, the first step is to obtain a matrix expression for μ. The way to do this is similar to the *method of second-order identities* [15,18,25], but our method avoids some unnecessary restrictions, broadening the range of application, within the scope of the problem we are studying.

Relationship of μ with the hidden Markov chain, using the matrix expression obtained, allows us to use the rich theory pertaining to these processes: the ergodic theorem, the Shannon–McMillan theorem, and the properties of Shannon entropy. We obtain a general result on the dimension of μ, expressing it as a function of an entropy, as occurs in the case with the open set condition.

Relationships between dimension and entropy have been obtained since the work of Besicovitch, Eggleston, and Billingsley [3,4,10] (see for instance [2,17,22,27]). We find a relationship of this type for the self-similar measures with overlapping which are studied.

We will obtain the absolute continuity or the singularity of μ from the elementary well-known result that characterizes the case of maximum entropy by means of the discrete uniform distribution.

Our results give lower and upper bounds for the dimension, which in some cases allow its calculation with high precision. Specifically, we obtain two sequences converging to the dimension, one of them non-increasing and the other non-decreasing.

This improves the results on the dimension of μ obtained in the cases considered in [15,18]. The general focus of those works is the multifractal spectrum of μ. One outstanding aspect of multifractal analysis is the study of dimension. A formula for the dimension is obtained in only some cases; this formula expresses dimension as a limit about which the only thing known is that it is converging (in one case a non-decreasing sequence is obtained). This limits its application in the study of the absolute continuity or singularity of μ and restricts the usefulness of the analysis from a computational point of view.

In Section 2, we review some notions and results on dimension, self-similar measures and Shannon entropy.

In Section 3, we realize the study of the dimension and the absolute continuity or singularity of μ.

In Section 4, we explain how to implement the calculations, we obtain absolute continuity for some cases studied in [15,18], and we exhibit figures for some of these measures.

2. Preliminaries

Let μ be a compactly supported finite Borel measure on $\mathbb{R}^d$. Consider the *spherical local dimension of μ at x*

$$\theta(\mu, x) = \lim_{r \to 0} \frac{\log \mu B(x, r)}{\log r},$$

where $B(x, r)$ denotes the closed ball centred at x and with radius r. The measure μ is called *exact dimensional* if for μ-a.e. (almost every) x the limit exists and takes a constant value. For an exact dimensional measure several notions of dimension coincide.

The *lower* and *upper Hausdorff dimensions of the measure μ* are defined as

$$\dim_* \mu = \inf\{\dim A\colon\ \mu A > 0\},$$

$$\dim^* \mu = \inf\bigl\{\dim A\colon\ \mu A^c = 0\bigr\},$$

where $\dim A$ is the Hausdorff dimension of the set A. The *lower* and *upper packing dimensions of μ* are defined in the same way, replacing $\dim A$ with the packing dimension of A (see [9, Definition 3.3.11 and Section 1.2]).

The measure μ is exact dimensional with $\theta(\mu, x) = \alpha$ for μ-a.e. x if and only if the four notions of dimension considered above coincide and take the value α (this follows easily from [9, Theorem 3.3.14]). This common value for an exact dimensional measure μ is called *dimension of μ* and it is denoted by $\dim \mu$. In what follows all the measures considered are exact dimensional; when reference is made to $\dim \mu$ it should be kept in mind that μ is always assumed to be exact dimensional.

If $\dim \mu = \alpha$ then the *entropy dimension* (or *Renyi dimension*) of μ is also equal to α. This will be mentioned again in Remark 4.

A sufficient condition for μ to be exact dimensional is the differentiability at $q = 1$ of the *L^q-spectrum of μ*. If $\tau'(1) = \alpha$ then $\dim \mu = \alpha$ (see [19]). In [15] the dimension of μ is studied using this result (in Corollaries 3.3 and 4.7 and Theorem 1.4).

We recall some notions pertaining to self-similar measures (see [11,12]). Let $\{\varphi_1, \ldots, \varphi_n\}$ be a *system of contractive similarities* on $\mathbb{R}^d$. Let E be the associated *self-similar set*, that is the unique non-empty compact set E such that $E = \bigcup_{i=1}^n \varphi_i E$. The system is said to satisfy the *open set condition* if there exists a non-empty open set U such that $\varphi_1(U), \ldots, \varphi_n(U)$ are disjoint and $\bigcup_{i=1}^n f_i(U) \subset U$. The system is *homogeneous* if all the contractivity ratios are equal.

Let $\{w_1, \ldots, w_n\}$ be a system of strictly positive probabilities (summing to one) and let μ be the *self-similar measure* associated to the *weighted* system of contractive similarities, that is the unique Borel probability measure with

$$\mu = \sum_{i=1}^n w_i \cdot \mu \circ \varphi_i^{-1};$$

it is supported on E. If $\{\varphi_1, \dots, \varphi_n\}$ satisfies the open set condition then

$$\dim \mu = \frac{\sum_{i=1}^n w_i \log w_i}{\sum_{i=1}^n w_i \log r_i}, \tag{1}$$

where $\{r_1, \dots, r_n\}$ are the contractivity ratios. See [9, Theorem 5.2.5] for a proof of this result, obtained by geometrical considerations and the strong law of large numbers. In [7, Theorem 1], a slightly more general result is proved. We note that $-\sum_{i=1}^n w_i \log_2 w_i$ is the Shannon entropy of the distribution given by $\{w_1, \dots, w_n\}$.

Here, we state some well-known facts about Shannon entropy (see [1,5,8]).

Let M be a finite set. Consider a stochastic process $V = (V_1, V_2, \dots)$ with values in M^∞ and the distribution given by

$$P\{V_1 = i_1, \dots, V_k = i_k\} = Q[i_1, \dots, i_k],$$

where Q is a probability measure on the product σ-algebra on M^∞ and

$$[i_1, \dots, i_k] = \{(j_1, j_2, \dots) \in M^\infty\colon\ (j_1, \dots, j_k) = (i_1, \dots, i_k)\}$$

is the cylinder in M^∞ with base $(i_1, \dots, i_k) \in M^k$.

The Shannon entropy of $(V_1, \dots, V_k)$ is defined as

$$H(V_1, \dots, V_k) = -\sum_{i_1, \dots, i_k \in M} Q[i_1, \dots, i_k] \log_2 Q[i_1, \dots, i_k].$$

The *Shannon entropy of* V, or of Q, is defined as

$$H(V) = \lim_{k \to \infty} \frac{1}{k} \cdot H(V_1, \dots, V_k) = H(Q) \tag{2}$$

if the limit exists.

The conditional Shannon entropy can be defined as

$$H(V_k / V_1, \dots, V_{k-1}) = H(V_1, \dots, V_k) - H(V_1, \dots, V_{k-1}).$$

If V is *stationary*, i.e., Q is shift-invariant, then the Shannon entropy of V exists and we have (see [8, Theorem 4.2.1])

$$H(V) = \lim_{k \to \infty} H(V_k / V_1, \dots, V_{k-1}). \tag{3}$$

The limit in (2) is the Cèsaro limit of the sequence in (3). If V is stationary then both sequences are non-increasing as well as converging, but the sequence in (3) converges more rapidly than the one in (2).

We can use (3) for estimating $H(V)$ for some processes V that we will associate to the self-similar measures under study. We will also obtain a converging non-decreasing sequence from the following result.

Let $X = (X_1, X_2, \dots)$ be a stationary Markov chain with finite state space S and let $\phi : S \to M$ be a relabelling of the states. Assume that V is the hidden Markov chain given by $V_k = \phi(X_k)$. From [6, Lemma 3.1] we know that the sequence $H(V_k / X_1, V_2, \dots, V_{k-1}) = H(X_1, V_2, \dots, V_k) - H(X_1, V_2, \dots, V_{k-1})$ is non-decreasing in k and

$$H(V) = \lim_{k \to \infty} H(V_k / X_1, V_2, \dots, V_{k-1}), \tag{4}$$

where

$$H(X_1, V_2, \dots, V_k) = -\sum_{j \in S} \sum_{(i_2, \dots, i_k) \in M^{k-1}} p(j, i_2, \dots, i_k) \log_2 p(j, i_2, \dots, i_k),$$

and $p(j, i_2, \dots, i_k) = P\{X_1 = j, V_2 = i_2, \dots, V_k = i_k\}$.

3. Homotheties with centres in a lattice and ratios $1/L$

Let $\{\varphi_1, \dots, \varphi_n\}$ be a homogeneous system of homotheties in $\mathbb{R}^d$ with centres $\mathbf{c}_1, \dots, \mathbf{c}_n$ in a lattice and ratio L^{-1} where $L \geqslant 2$ is an integer. Let E be the associated self-similar set.

Let C be the convex hull of $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$. Since the φ_i are homotheties, we have $\varphi_i C \subset C$, and from [12, Observation 3.1(8)] we have $E \subset C$. Since $\{\mathbf{c}_1, \dots, \mathbf{c}_n\} \subset E$, it follows that the convex hull of E coincides with C.

Let $\{w_1, \dots, w_n\}$ be a system of strictly positive probabilities, summing to one, and let μ be the unique Borel probability measure satisfying $\mu = \sum_{i=1}^n w_i \cdot \mu \circ \varphi_i^{-1}$.

Definition 1. Such a probability measure μ will be called a homogeneous rational self-similar measure.

The class of homogeneous rational self-similar measures includes cases with the open set condition, for which $\dim\mu$ is known (see (1)), and thus the cases of interest to us are those for which this condition is not satisfied.

In the case when $d=1$, we write c_i for the centres, without bold type.

In [23, Section 3] we analyse the particular case with $d=1$, $n=3$, $c_1=0$, $c_2=1/2$, $c_3=1$, $L=2$ and $w_1=w_2=w_3=1/3$.

It is easy to see that if f is any affine map on $\mathbb{R}^d$ then $f\varphi_i f^{-1}$ is a homothety having ratio L^{-1}, as is φ_i. The self-similar measure for the weighted i.f.s. $\{(f\varphi_i f^{-1}, w_i)\}$ is the pull back measure under f of the self-similar measure μ for the original weighted i.f.s. $\{(\varphi_i, w_i)\}$, as can be deduced directly from the definitions. Taking f to be an affine map such that $f(\mathbf{c}_i)\in\mathbb{Z}^d$ for all $i=1,\dots,n$, the new measure is bi-Lipschitz equivalent to μ, the contraction ratio is the same, and the centres of the homotheties become points with integer coordinates, so we can study this new measure instead. We assume from now on that

$$\{\mathbf{c}_1,\dots,\mathbf{c}_n\}\subset\mathbb{Z}^d.$$

We write $\varphi_{i_1,\dots,i_k}$ for $\varphi_{i_1}\circ\cdots\circ\varphi_{i_k}$.

Remark 1. These systems satisfy the finite type condition (see [20]): We have $\varphi_l(\mathbf{x})=L^{-1}\mathbf{x}+\mathbf{c}_l(1-L^{-1})$ and $\varphi_{i_1,\dots,i_k}(\mathbf{x})=L^{-k}\mathbf{x}+\sum_{m=1}^{k}L^{-(m-1)}\mathbf{c}_{i_m}(1-L^{-1})$, and hence $L^k\varphi_{i_1,\dots,i_k}(\mathbf{0})=\sum_{m=1}^{k}L^{k-m}\mathbf{c}_{i_m}(L-1)$. From this it follows that $L^k(\varphi_{i_1,\dots,i_k}(\mathbf{0})-\varphi_{j_1,\dots,j_k}(\mathbf{0}))$ has integer coordinates and, from this, the finite type condition can be checked.

3.1. A matrix expression for the self-similar measure

3.1.1. L-adic d-dimensional cubes

For $k=0,1,\dots,$ let J_k be the class of closed intervals

$$J_k=\bigl\{\bigl[(i-1)\cdot L^{-k}, i\cdot L^{-k}\bigr]:\ i\in\mathbb{Z}\bigr\}$$

and let $\mathfrak{D}_k$ be the class of cubes in $\mathbb{R}^d$ that are Cartesian products of elements in J_k. The $\mathfrak{D}_k$'s are nested and $\mathfrak{D}_{k+1}$ can be obtained from $\mathfrak{D}_k$ by dividing each element in L^d pieces, with L equal parts for each coordinate. Since the centres $\mathbf{c}_i$ have integer coordinates they are vertices of sets in $\mathfrak{D}_k$ for all k.

Remark 2. It is immediate that $\varphi_l(\mathfrak{D}_{k-1})=\mathfrak{D}_k$ and $\varphi_l^{-1}(\mathfrak{D}_k)=\mathfrak{D}_{k-1}$ for all k and l.

For $k\geqslant 0$ let

$$A_k=\bigcup\{F_1\cap F_2:\ F_1,F_2\in\mathfrak{D}_k,\ \ F_1\neq F_2\}$$

be the set of overlaps in $\mathfrak{D}_k$, which coincides with the union of the boundaries of the cubes in $\mathfrak{D}_k$. We have $\varphi_l^{-1}(A_k)=A_{k-1}\subset A_k$ for all k,l. We show that $\mu A_k=0$, and to do so we use the following lemma.

Lemma 1. *Let μ be the self-similar measure associated to a weighted system of contractive similarities $\{(\varphi_i,p_i):\ i=1,\dots,n\}$ in $\mathbb{R}^d$, possibly having overlaps. If A is not dense in the self-similar set E and $\bigcup_{j=1}^{n}\varphi_j^{-1}(A)\subset A$ then $\mu A=0$.*

Proof. The proof is a variant of an argument in the proof of [17, Theorem 2.1], but a complete proof is given for the convenience of the reader.

Let $I=\{1,\dots,n\}$ and $I^*=\bigcup_{k=1}^{\infty}I^k$.

Consider the measurable function $\pi: I^\infty\to E$ given by

$$\bigl\{\pi(i_1,i_2,\dots)\bigr\}=\bigcap_{k=1}^{\infty}\varphi_{i_1,\dots,i_k}(E).$$

It is well known that $\pi(I^\infty)=E$. Given an $x\in E$ and an $\mathbf{i}=(i_1,i_2,\dots)\in I^\infty$ such that $\pi(\mathbf{i})=x$, the shift $\mathbf{i}$-orbit of x is defined as the set $\gamma_{\mathbf{i}}(x)=\{x,x_1,x_2,\dots\}$, where $x_m=\varphi_{i_m}^{-1}\circ\cdots\circ\varphi_{i_1}^{-1}(x)\in E$ for $m\geqslant 1$, and the shift orbit of x is defined as $O(x)=\bigcup\{\gamma_{\mathbf{i}}(x):\ \mathbf{i}\in I^\infty,\pi(\mathbf{i})=x\}$.

For $\mathbf{i}\in I^\infty$ and $\mathbf{j}=(j_1,\dots,j_l)\in I^*$ let

$$\delta_{\mathbf{j}}(\mathbf{i})=\lim_{k\to\infty}\frac{1}{k}\operatorname{card}\bigl\{q:\ (i_q,i_{q+1},\dots,i_{q+l-1})=\mathbf{j},\ \ 1\leqslant q\leqslant k-l+1\bigr\}.$$

We consider the set

$$\mathcal{B}=\bigl\{\mathbf{i}\in I^\infty:\ \delta_{\mathbf{j}}(\mathbf{i})=p_{\mathbf{j}},\ \mathbf{j}\in I^*\bigr\},$$

where $p_{\mathbf{j}}=p_{j_1}\cdots p_{j_l}$ if $\mathbf{j}=(j_1,\dots,j_l)$.

For $\mathbf{i} \in \mathcal{B}$, $\mathbf{j} \in I^*$ we have $(i_q, i_{q+1}, \ldots, i_{q+l-1}) = \mathbf{j}$ infinitely often and hence for at least one q, since $p_{\mathbf{j}} > 0$. From this it follows that $O(\pi(\mathbf{i})) \cap \varphi_{\mathbf{j}}(E) \neq \emptyset$, and hence $O(\pi(\mathbf{i}))$ is dense in E for each $\mathbf{i} \in \mathcal{B}$, since $E = \bigcup_{\mathbf{t} \in I^k} \varphi_{\mathbf{t}}(E)$ with $\varphi_{\mathbf{t}} = \varphi_{t_1,\ldots,t_k}$.

Let ν be the product probability measure on I^∞ defined from the probability measure on I given by $(p_i\colon\ i = 1, \ldots, n)$. From the ergodic theorem we have $\nu(\mathcal{B}) = 1$, since I^* is countable. From this it follows that $\nu(D) = 0$, with $D = \{\mathbf{i} \in I^\infty\colon\ O(\pi(\mathbf{i}))$ is not dense in $E\}$.

For each $x \in A$ we have $O(x) \subset A$, since $\bigcup_{j=1}^n \varphi_j^{-1}(x) \in \bigcup_{j=1}^n \varphi_j^{-1}(A) \subset A$, and hence $O(x)$ is not dense in E since A is not. Therefore $\pi^{-1}A \subset D$ and hence $\mu A = \nu(\pi^{-1}A) \leqslant \nu(D) = 0$, since $\mu = \nu\pi^{-1}$. □

By passing to a lower dimensional affine subspace if needed, we will from now on assume that E is not contained in any $(d-1)$-dimensional affine subspace. In particular, $E \cap A_k^c \neq \emptyset$ for all $k \geqslant 0$.

Proposition 2. *For $k \geqslant 0$ we have $\mu A_k = 0$.*

Proof. Let $k \geqslant 0$. Since $E \cap A_k^c \neq \emptyset$ the closed set A_k is not dense in E. Since also $\bigcup_{l=1}^n \varphi_l^{-1} A_k \subset A_k$ the result follows from Lemma 1. □

3.1.2. Basic cubes

Since the support of μ is the compact set E there are only a finite number of cubes in $\mathfrak{D}_0$ having positive measure. Let $\langle 1\rangle, \ldots, \langle N\rangle$ be these cubes given in an arbitrary fixed ordering. Let

$$M = \{1, \ldots, L^d\}.$$

Each cube $\langle j\rangle$, $1 \leqslant j \leqslant N$, splits into L^d cubes in $\mathfrak{D}_1$. We label these cubes as $\langle j; i_1\rangle$, $i_1 \in M$, with the following ordering. We consider for each of these cubes the point with minimal coordinates, and we order the cubes according to the lexicographical ordering of these points.

We proceed further with the labelling. Each cube $\langle j; i_1\rangle$ splits into L^d cubes in $\mathfrak{D}_2$, which we label as $\langle j; i_1, i_2\rangle$, $i_2 \in M$, with the same criterion, and we iterate this process, thus labelling as $\langle j; \mathbf{i}\rangle = \langle j; i_1, \ldots, i_k\rangle$, for $\mathbf{i} = (i_1, \ldots, i_k) \in M^k$, all cubes in $\mathfrak{D}_k$ into which the primary cubes $\langle 1\rangle, \ldots, \langle N\rangle$ split.

We note that although $\mu\langle i\rangle > 0$ for all i, it can happen that $\mu\langle i; \mathbf{i}\rangle = 0$ for some k and $\mathbf{i} \in M^k$. However, if $\mu D > 0$ for some $D \in \mathfrak{D}_k$ then D is contained in a set $\langle i\rangle$ (a cube) and hence is a set $\langle i; \mathbf{i}\rangle$; we use this fact in the proof of Lemma 3(1).

Lemma 3. *We have*

(1) *Each $\varphi_l\langle j\rangle$ is a set $\langle i; m\rangle$.*
(2) *We have $\varphi_l\langle j\rangle = \langle i; m\rangle$ if and only if $\varphi_l\langle j; \mathbf{i}\rangle = \langle i; m, \mathbf{i}\rangle$ for all $k \geqslant 0$ and $\mathbf{i} \in M^k$.*
(3) *If $\varphi_l^{-1}\langle i; m, \mathbf{i}\rangle$ is not a set $\langle j; \mathbf{i}\rangle$ for any j then $\mu(\varphi_l^{-1}\langle i; m, \mathbf{i}\rangle) = 0$.*

Proof. (1) We have $\varphi_l\langle j\rangle \in \mathfrak{D}_1$ and $\mu(\varphi_l\langle j\rangle) = \sum_{t=1}^n w_t \cdot \mu(\varphi_t^{-1}\varphi_l\langle j\rangle) \geqslant w_l \cdot \mu\langle j\rangle > 0$.

(2) This is obvious since φ_l is a similarity.

(3) In spite of Lemma 3(1), although $\varphi_l^{-1}\langle i; m\rangle \in \mathfrak{D}_0$ it can be that it is not a set $\langle j\rangle$. If $\varphi_l^{-1}\langle i; m\rangle \neq \langle j\rangle$ for $j = 1, \ldots, N$, then if it intersects some set $\langle j\rangle$ it does so in the boundary of $\langle j\rangle$, and hence, by Proposition 2, we have $\mu(\varphi_l^{-1}\langle i; m\rangle) = 0$. The result follows from Lemma 3(2) since φ_l is bijective. □

3.1.3. Matrix expression for the measure of the basic cubes

For $m \in M$ let Z_m be the $N \times N$ matrix with

$$Z_m(i, j) = w_l \quad \text{if } \varphi_l\langle j\rangle = \langle i; m\rangle \text{ for some } l,$$

and $Z_m(i, j) = 0$ otherwise. Of course, for given m, i, j there is at most one such l if the φ_l are all different, which we assume.

From Lemma 3(2) and since φ_l is bijective, we have

$$Z_m(i, j) = w_l \quad \text{if } \varphi_l^{-1}\langle i; m, \mathbf{i}\rangle = \langle j; \mathbf{i}\rangle,$$

and in this case

$$w_l \cdot \mu\big(\varphi_l^{-1}\langle i; m, \mathbf{i}\rangle\big) = Z_m(i, j) \cdot \mu\langle j; \mathbf{i}\rangle.$$

From this and Lemma 3(3) we have

$$\mu\langle i; m, \mathbf{i}\rangle = \sum_{l=1}^n w_l \cdot \mu\big(\varphi_l^{-1}\langle i; m, \mathbf{i}\rangle\big) = \sum_{j=1}^N Z_m(i, j) \cdot \mu\langle j; \mathbf{i}\rangle. \tag{5}$$

Let $\mathbf{e}_j$ be the row N-vector with 1 in the jth entry and zero elsewhere. For $\mathbf{i} \in M^k$ we consider the column vector $\mu\langle\cdot;\mathbf{i}\rangle = (\mu\langle j;\mathbf{i}\rangle\colon\ j = 1,\dots,N)^t$, so that

$$\mu\langle j;\mathbf{i}\rangle = \mathbf{e}_j\mu\langle\cdot;\mathbf{i}\rangle. \tag{6}$$

The relation obtained in (5) can be expressed as

$$\mu\langle\cdot;m,\mathbf{i}\rangle = Z_m\mu\langle\cdot;\mathbf{i}\rangle. \tag{7}$$

We call $\mathbf{p}^t = \mu\langle\cdot\rangle = (\mu\langle 1\rangle,\dots,\mu\langle N\rangle)^t$.

Proposition 4. *For $j \in \{1,\dots,N\}$, $k \geqslant 0$ and $\mathbf{i} \in M^k$ we have*

$$\mu\langle j;\mathbf{i}\rangle = \mathbf{e}_j Z_{\mathbf{i}}\mathbf{p}^t,$$

with $Z_{\mathbf{i}} = Z_{i_1}\cdots Z_{i_k}$ if $\mathbf{i} = (i_1,\dots,i_k)$.

Proof. From (7) we obtain $\mu\langle\cdot;\mathbf{i}\rangle = Z_{\mathbf{i}}\mu\langle\cdot\rangle = Z_{\mathbf{i}}\mathbf{p}^t$ and the result follows from (6). □

It is easy to obtain a formula for computing the matrices Z_m. We do it for $d = 1$; for $d > 1$ the formula can be obtained by considering an expression with d coordinates for m, i, j and $\mathbf{c}_l$.

For simplicity we assume that $\min_{1\leqslant l\leqslant n} c_l = 0$ (and hence $\langle 1\rangle = [0,1]$). We consider the closed intervals $I(j) = [j-1, j]$ for $j = 1,\dots,\max_{1\leqslant l\leqslant n} c_l$. Note that each $\langle i\rangle$ must be a set $I(j)$, but some of the sets $I(j)$ can have null measure and hence not be sets $\langle i\rangle$.

We have $\varphi_l(x) = c_l + (x - c_l)/L$ and

$$\varphi_l I(j) = \left[c_l + \frac{j-1-c_l}{L}, c_l + \frac{j-1-c_l}{L} + \frac{1}{L}\right]. \tag{8}$$

We have

$$\frac{j-1-c_l}{L} = \mathrm{floor}\left(\frac{j-1-c_l}{L}\right) + \mathrm{frac}\left(\frac{j-1-c_l}{L}\right),$$

where "floor" is "the greatest integer less than or equal to" and $\mathrm{frac}(z) = z - \mathrm{floor}(z)$.

We now consider, for $j = 1,\dots,\max_{1\leqslant l\leqslant n} c_l$ and $m = 1,\dots,L$, the closed intervals $I(j,m) = [j-1+(m-1)/L, j-1+m/L]$, so that if $I(j) = \langle i\rangle$ then $I(j,m) = \langle i;m\rangle$.

It is easy to check that for given l, j there are a unique i and a unique m with $\varphi_l I(j) = I(i,m)$. From (8) we obtain $i - 1 = c_l + \mathrm{floor}(\frac{j-1-c_l}{L})$ and $\frac{m-1}{L} = \mathrm{frac}(\frac{j-1-c_l}{L})$, and hence these i, m are

$$i = c_l + 1 + \mathrm{floor}\left(\frac{j-1-c_l}{L}\right), \tag{9}$$

$$m = j - c_l - L\cdot\mathrm{floor}\left(\frac{j-1-c_l}{L}\right). \tag{10}$$

Assume that $\langle i\rangle = I(t_i)$ for $i = 1,\dots,N$. We have $Z_m(i,j) = w_l$ if $\varphi_l I(t_j) = I(t_i,m)$, and hence we can obtain the matrices Z_m from (9) and (10).

In Proposition 7 we will give a method to calculate $\mathbf{p}^t$, and in Remark 3 we will give a procedure to identify the sets $\langle i\rangle = I(t_i)$, which are those $I(j)$ having positive measure.

In [25] a matrix expression is obtained for the self-similar measure associated to Bernoulli convolutions for which the ratio is the inverse of the golden number. In [15,18] the *method of second-order identities* is developed based on the ideas of [25], obtaining a matrix expression for some self-similar measures. The expression obtained here is similar. It does not cover the case considered in [25], but does cover a class that is wider than those studied in [15,18].

In addition, the study of some elements of the matrix expression that we make in Section 3.2 will allow us in Section 3.3 to relate μ to a hidden Markov chain and, from this, to obtain some results on the dimension of μ.

3.2. Properties of the matrix Z

Recall that $M = \{1,\dots,L^d\}$. Let

$$Z = \sum_{m\in M} Z_m.$$

We begin with a lemma that is useful in showing that Z is irreducible.

Lemma 5. *Let μ be the self-similar measure associated to a weighted system of contractive similarities, possibly with overlaps. If D is an open set with $\mu D > 0$ then for μ-a.e. x there are $k \geqslant 1$ and $l_1, \dots, l_k \in \{1, \dots, n\}$ with $\varphi_{l_k}^{-1} \circ \cdots \circ \varphi_{l_1}^{-1}(x) \in D$.*

The proof of Lemma 5 is a part of the proof of Morán and Rey [17, Theorem 2.1]. This theorem states, under the open set condition, that the overlaps have null measure. In its proof it is shown that the *orbits* of a.e. $x \in E$ are dense in $E = \operatorname{supp} \mu$. We note that the proof of this result about orbits remains valid also without the open set condition (see the proof of Lemma 1). It is easy to see that this result is equivalent to Lemma 5.

Proposition 6. *Z is an irreducible matrix and its transpose is a stochastic matrix, and hence its greatest eigenvalue is one and it is simple.*

Proof. Let $i, j \in \{1, \dots, N\}$. By Proposition 2, the interior of $\langle j \rangle$ is an open set with measure equal to that of $\langle j \rangle$, and hence is positive like that of $\langle i \rangle$. It follows from Lemma 5 that there is an x in the interior of $\langle i \rangle$ with $\varphi_{l_k}^{-1} \circ \cdots \circ \varphi_{l_1}^{-1}(x) \in \langle j \rangle$ for some $k \geqslant 1$ and $l_1, \dots, l_k \in \{1, \dots, n\}$. Since $\varphi_{l_1} \circ \cdots \circ \varphi_{l_k} \langle j \rangle$ is a set $\langle t; i_1, \dots, i_k \rangle$ that contains x it must be $t = i$, and hence

$$\varphi_{l_1} \circ \cdots \circ \varphi_{l_k} \langle j \rangle = \langle i; i_1, \dots, i_k \rangle \tag{11}$$

for some $i_1, \dots, i_k$.

Let $t_0 = j$ and $t_k = i$. It is easy to check from (11) and Lemma 3 ((1) and (2)) that there are $t_1, \dots, t_{k-1}$ with $\varphi_{l_k} \langle t_0 \rangle = \langle t_1; i_k \rangle$ and

$$\begin{aligned} \varphi_{l_{k-u}} \circ \varphi_{l_{k-u+1}} \circ \cdots \circ \varphi_{l_k} \langle j \rangle &= \varphi_{l_{k-u}} \langle t_u; i_{k-u+1}, \dots, i_k \rangle \\ &= \langle t_{u+1}; i_{k-u}, i_{k-u+1}, \dots, i_k \rangle \end{aligned}$$

for $u = 1, \dots, k-1$. From this and Lemma 3(2) we obtain $Z_{i_{k-u}}(t_{u+1}, t_u) = w_{l_{k-u}} > 0$ for $u = 0, \dots k-1$, and hence the $(i, j) = (t_k, t_0)$ entry of the matrix power Z^k satisfies

$$Z^k(i, j) \geqslant Z_{i_1}(t_k, t_{k-1}) Z_{i_2}(t_{k-1}, t_{k-2}) \cdots Z_{i_k}(t_1, t_0) > 0.$$

Therefore Z is irreducible.

By Lemma 3(1), $\varphi_l \langle j \rangle$ is a set $\langle i; m \rangle$ for each l, j, and from this it follows, for $j = 1, \dots, N$, that the column j of Z sums to $w_1 + \cdots + w_n = 1$. Therefore the transpose of Z is a stochastic matrix.

Applying the Perron–Frobenius theorem completes the proof. □

Proposition 7. *The unique probability vector $\mathbf{x}$ solving $Z\mathbf{x} = \mathbf{x}$ is $\mathbf{p}^t$.*

Proof. By Proposition 2 we have $\mu A_0 = 0$, and thus the overlaps of the sets $\langle i \rangle$ have null measure. Since $E = \operatorname{supp} \mu \subset \bigcup_{i=1}^N \langle i \rangle$ and $\mu E = 1$ it follows that $\mathbf{p}^t$ is a probability vector.

From (7) and since $\mu A_1 = 0$ we have

$$Z\mathbf{p}^t = \sum_{m \in M} Z_m \mu \langle \cdot \rangle = \sum_{m \in M} \mu \langle \cdot; m \rangle = \mu \langle \cdot \rangle = \mathbf{p}^t.$$

By Proposition 6, one is a simple eigenvalue of Z which proves the uniqueness. □

Remark 3. This result allows us to make explicit calculations for $\mathbf{p}^t$ and then for the $\mu\langle j; \mathbf{i} \rangle$'s by means of Proposition 4. It can also be used to identify the sets $\langle i \rangle$. If we had incorrectly taken a set in $\mathfrak{D}_0$ as if it were a set $\langle i \rangle$, then the null value of its measure would be revealed by the null value in the corresponding entry of the resulting (incorrect) $\mathbf{p}^t$.

3.3. Dimension of the self-similar measure

We show that for μ-a.e. x the local dimension $\theta(\mu, x)$ exists and does not depend on x, and its value is the Shannon entropy of an associated ergodic measure on M^∞ divided by $\log_2 L$. This gives the Hausdorff and packing dimensions of μ. From this we also deduce some conditions for the absolute continuity of μ (w.r.t. Lebesgue measure).

We write $\nu_1 \ll \nu_2$ for the absolute continuity of a measure ν_1 w.r.t. a measure ν_2, and $\mathcal{L}$ for the d-dimensional Lebesgue measure.

We construct an auxiliary measure η by considering the restrictions of μ to the cubes $\langle j \rangle$, translating them to a given fixed cube and piling the restrictions up together.

Let $\langle 0 \rangle$ be the unit cube $[0, 1]^d$ and, for $j = 1, \dots, N$, let g_j be the translation with $g_j \langle 0 \rangle = \langle j \rangle$. For $\mathbf{i} \in M^k$ we call $g_j^{-1} \langle j; \mathbf{i} \rangle$ by $\langle 0; \mathbf{i} \rangle$; this is a subset of $\langle 0 \rangle$ which does not depend on j. We consider the Borel measure $\eta = \sum_{j=1}^N \eta_j$ with

$\eta_j = \mu_j \circ g_j$, where $\mu_j(\cdot) = \mu(\langle j\rangle \cap \cdot)$ is the restriction of μ to $\langle j\rangle$. The measure η is concentrated on $\langle 0\rangle$. It is a probability measure since μ is and the overlaps of the sets $\langle j\rangle$ have null μ-measure. We have

$$\eta\langle 0; \mathbf{i}\rangle = \sum_{j=1}^{N} \mu\langle j; \mathbf{i}\rangle. \tag{12}$$

We first show that, if $\dim \eta = \alpha$ (recall that according to our convention this presupposes that η is exact dimensional) then $\dim \mu = \alpha$.

From [16, Lemma 2.4], if $\dim \eta = \alpha$ then $\dim \eta_j = \alpha$, since $\eta_j \ll \eta$. Since g_j is bi-Lipschitz we have $\dim \mu_j = \dim \eta_j = \alpha$ for $j = 1, \ldots, N$. Hence $\theta(\mu_j, x) = \alpha$ for μ_j-a.e. x. Since $\mu A_0 = 0$ (see Proposition 2) and $\mu = \sum_{j=1}^{N} \mu_j$, it follows that $\theta(\mu, x) = \alpha$ for μ-a.e. x and hence $\dim \mu = \alpha$.

We next show that η is in fact exact dimensional and we obtain $\dim \eta$, thus obtaining $\dim \mu$ and proving Theorem 9 below.

Let Q be the measure on the product σ-algebra on M^∞ given by

$$Q\,[\mathbf{i}] = \mathbf{e} Z_{\mathbf{i}} \mathbf{p}^t \quad \text{for } k \geqslant 1,\ \mathbf{i} \in M^k,$$

where $[\mathbf{i}]$ is the cylinder in M^∞ with base $\mathbf{i}$ and $\mathbf{e} = \sum_{j=1}^{N} \mathbf{e}_j$ is the row N-vector with every entry 1. Note that from (12) and Proposition 4 we have $Q\,[\mathbf{i}] = \eta\langle 0; \mathbf{i}\rangle$.

In [23, Proposition 1] it is shown that Q is a shift-invariant and ergodic probability measure. This follows from the following facts: $Z = \sum_{m\in M} Z_m$ is irreducible, its greatest eigenvalue is one, $\mathbf{e}Z = \mathbf{e}$, $Z\mathbf{p}^t = \mathbf{p}^t$, and $\mathbf{e}\mathbf{p}^t = 1$. Furthermore, in [23, Proposition 1] it is shown that Q is the distribution of an ergodic hidden Markov chain, which we denote by $V = V(Z_m\colon m = 1, \ldots, L^d) = (V_1, V_2, \ldots)$. By the theorem of Shannon–McMillan (see [8, Theorem 15.7.1]) we have

$$\lim_{k\to\infty} -\frac{1}{k} \log_2 Q\,[i_1, \ldots, i_k] = H \tag{13}$$

for Q-a.e. $(i_1, i_2, \ldots) \in M^\infty$, where H is the Shannon entropy of Q.

Let $A = \bigcup_{k=0}^{\infty} A_k$. Then $\mu A = 0$, since $\mu A_k = 0$ for all k (see Proposition 2). Since $g_j(A_k) = A_k$ for all j, k, it follows that the overlaps of the sets $\langle 0; i_1, \ldots, i_k\rangle$ for given k are contained in A_k, and we also have $\eta A_k = 0$ and hence $\eta A = 0$.

For $x \in \langle 0\rangle \cap A^c$ and $k \geqslant 0$ there is a unique set $\langle 0; i_1, \ldots, i_k\rangle$ that contains x; let $D_k(x)$ be this set. There holds $D_{k+1}(x) \subset D_k(x)$ for η-a.e. x and all k. Notice that $\bigcap_{k=1}^{\infty} D_k(x) = \{x\}$.

Proposition 8. *We have*

$$\lim_{k\to\infty} -\frac{1}{k} \log_2 \eta D_k(x) = H \quad \textit{for } \eta\textit{-a.e. } x \in \langle 0\rangle.$$

Proof. Consider the measurable function $\pi : M^\infty \to \langle 0\rangle$ given by

$$\bigl\{\pi(i_1, i_2, \ldots)\bigr\} = \bigcap_{k=1}^{\infty} \langle 0; i_1, \ldots, i_k\rangle,$$

so that $\pi[\mathbf{i}] = \langle 0; \mathbf{i}\rangle$ for $k \geqslant 0$, $\mathbf{i} \in M^k$. Since $Q\,[\mathbf{i}] = \eta\langle 0; \mathbf{i}\rangle$ we have $Q = \eta\pi$.

Let $\Gamma \subset M^\infty$ be the set where (13) holds, with $Q\,\Gamma = 1$. Let $B = (\pi\,\Gamma) \cap A^c$. Since $B \subset \langle 0\rangle \cap A^c$, $\pi^{-1}(x)$ has cardinality one for $x \in B$. It is easily seen that $\pi^{-1}B \subset \Gamma$ and $\eta B = 1$.

Let $x \in B$ and $(i_1, i_2, \ldots) = \pi^{-1}(x) \in \Gamma$. We have $D_k(x) = \langle 0; i_1, \ldots, i_k\rangle$ and hence

$$\eta D_k(x) = \eta\langle 0; i_1, \ldots, i_k\rangle = \eta\pi[i_1, \ldots, i_k] = Q\,[i_1, \ldots, i_k].$$

The proposition follows from (13), since $\eta B = 1$. $\square$

From this we find that the *cylindrical local dimension* of η is

$$\lim_{k\to\infty} \frac{\log \eta D_k(x)}{\log|D_k(x)|} = \lim_{k\to\infty} \frac{\log_2 \eta D_k(x)}{\log_2(\sqrt{d}\cdot L^{-k})} = \frac{H}{\log_2 L} \quad \text{for } \eta\text{-a.e. } x \in \langle 0\rangle,$$

where $|D_k(x)|$ is the diameter of $D_k(x)$. From [22, Theorem 15.3] we find that the spherical local dimension $\theta(\eta, x)$ is also $H/\log_2 L$ for η-a.e. x. Hence $\dim \eta = H/\log_2 L$ and, finally, since $\dim \mu = \dim \eta$ we have:

Theorem 9. *Let μ be a homogeneous rational self-similar measure as in Definition* 1. *Let $V = V(Z_m\colon m = 1, \ldots, L^d)$ be the associated ergodic hidden Markov chain as above, and let H be its Shannon entropy. Then μ is an exact dimensional measure with*

$$\dim \mu = \frac{H}{\log_2 L}.$$

Since $H \leqslant H(V_1)$ we have immediately the following estimate:

$$\dim \mu \leqslant \frac{H(V_1)}{\log_2 L} = -\frac{1}{\log L} \sum_{m \in M} \left(\mathbf{e} Z_m \mathbf{p}^t\right) \log\left(\mathbf{e} Z_m \mathbf{p}^t\right).$$

Remark 4. The entropy dimension of μ is defined as

$$\dim_e \mu = \lim_{\delta \to 0} \frac{H(\mu, \delta)}{-\log \delta},$$

where

$$H(\mu, \delta) = \inf\left\{H(\mu, \mathcal{P}):\ \mathcal{P} \text{ is a finite Borel partition of } \operatorname{supp} \mu,\ |\mathcal{P}| \leqslant \delta\right\},$$

$H(\mu, \mathcal{P}) = -\sum_{A \in \mathcal{P}} (\mu A) \log(\mu A)$, and $|\mathcal{P}|$ is the maximum of the diameters of elements of $\mathcal{P}$. Taking $\mathcal{P}_k = \{\langle j; \mathbf{i}\rangle:\ j = 1, \ldots, N, \mathbf{i} \in M^k\}$ for $\delta = \sqrt{d} L^{-k} = |\mathcal{P}_k|$ we have

$$\frac{H(\mu, \mathcal{P}_k)}{-\log \delta} = -\frac{1}{k \log L - \log \sqrt{d}} \sum_{j=1}^{N} \sum_{\mathbf{i} \in M^k} \mu\langle j; \mathbf{i}\rangle \log \mu\langle j; \mathbf{i}\rangle \tag{14}$$

(with $0 \log 0 = 0$). By [27, Theorem 4.4] we know that if $\dim \mu = \alpha$ then $\dim_e \mu = \alpha$. From Theorem 9 and (2) (in Section 2) we have

$$\dim_e \mu = \dim \mu = \lim_{k \to \infty} -\frac{1}{k \log L} \sum_{\mathbf{i} \in M^k} Q[\mathbf{i}] \log Q[\mathbf{i}]. \tag{15}$$

This choice of $\mathcal{P}_k$ brings about something similar to (14) with the sequence in (15); recall that $Q[\mathbf{i}] = \sum_{j=1}^{N} \mu\langle j; \mathbf{i}\rangle$.

3.4. Absolute continuity and singularity

Using Theorem 9 and the well-known result on Shannon entropy given in (16) and (17) we next obtain results on absolute continuity for homogeneous rational self-similar measures.

It is easy to check that the maximum entropy of a finite distribution is uniquely attained by the uniform distribution. In our case we have

$$H(V_1, \ldots, V_k) \leqslant \log_2\left(L^{kd}\right), \tag{16}$$

with equality if and only if

$$P\left\{(V_1, \ldots, V_k) = \mathbf{i}\right\} = L^{-kd} \quad \text{for all } \mathbf{i} \in M^k, \tag{17}$$

where $P\{(V_1, \ldots, V_k) = \mathbf{i}\} = Q[\mathbf{i}] = \mathbf{e} Z_{\mathbf{i}} \mathbf{p}^t$.

The dimension of the self-similar measure μ is less than or equal to the dimension d of $\mathbb{R}^d$. We next show that equality holds if and only if μ is absolutely continuous. We give other characterizations that will facilitate further analysis of this case.

Proposition 10. *The following properties are equivalent*:

(1) $\mu \ll \mathcal{L}$.
(2) $\dim \mu = d$.
(3) $\mathbf{e} Z_{\mathbf{i}} \mathbf{p}^t = L^{-kd}$ *for all* k *and* $\mathbf{i} \in M^k$.
(4) η *is the Lebesgue measure on* $[0, 1]^d$.
(5) μ *is not singular.*

Proof. (1) $\Rightarrow$ (2) Obvious from the definitions.

(2) $\Leftrightarrow$ (3) From Theorem 9 we have $\dim \mu = H / \log_2 L$, and hence $\dim \mu = d$ if and only if $H = \log_2(L^d)$. From the properties given in (16) and (17) we have $H = \log_2(L^d)$ if and only if equality holds in (16) for all k, since the sequence $\{(1/k) \cdot H(V_1, \ldots, V_k)\}_k$ is non-increasing and converges to H.

(3) $\Rightarrow$ (4) We have $\eta\langle 0; \mathbf{i}\rangle = \sum_{j=1}^{N} \mu\langle j; \mathbf{i}\rangle = \mathbf{e} Z_{\mathbf{i}} \mathbf{p}^t = L^{-kd}$ for all k and $\mathbf{i} \in M^k$. The result follows by a standard argument.

(4) $\Rightarrow$ (1) Since $\mu_j \circ g_j \ll \eta$ and $\eta \ll \mathcal{L}$ we have $\mu_j \circ g_j \ll \mathcal{L}$, and since g_j is a translation we have $\mu_j \ll \mathcal{L}$, for $j = 1, \ldots, N$. Finally, since $\mu = \sum_{j=1}^{N} \mu_j$ we have $\mu \ll \mathcal{L}$.

(5) $\Leftrightarrow$ (1) It is well known that arbitrary self-similar measures are either absolutely continuous or singular with respect to Lebesgue measure. □

When μ is singular, the condition in Proposition 10(3) usually fails for $k = 1$. We have found only one case for which verifying the singularity of μ required checking that Proposition 10(3) fails for $k = 2$. This was the case mentioned above in [23]. We have not found any case for which we had to check this condition for $k > 2$. All the cases we tried that did not satisfy the easy-to-check hypothesis of Corollary 12 below are like this.

In the following two corollaries, we provide sufficient conditions for absolute continuity. The second is a simple condition for the initial elements of the problem and the first for the matrices Z_m.

Corollary 11. *If $\mathbf{e}Z_m = L^{-d}\mathbf{e}$ for all $m \in M$, or $Z_m\mathbf{p}^t = L^{-d}\mathbf{p}^t$ for all $m \in M$, then $\mu \ll \mathcal{L}$.*

Proof. If $\mathbf{e}Z_m = (L^d)^{-1}\mathbf{e}$ for all $m \in M$ then $\mathbf{e}Z_{\mathbf{i}} = (L^d)^{-k}\mathbf{e}$ and hence $\mathbf{e}Z_{\mathbf{i}}\mathbf{p}^t = (L^d)^{-k}\mathbf{e}\mathbf{p}^t = L^{-kd}$ for all k and $\mathbf{i} \in M^k$. Thus the condition in Proposition 10(3) holds. The proof of the other claim is similar. $\square$

For $\mathbf{j} = (j_1, \dots, j_d) \in I := \{0, \dots, L-1\}^d$ let

$$J(L, \mathbf{j}) = \bigl\{\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{Z}^d\colon\ z_i \pmod L = j_i,\ i = 1, \dots, d\bigr\} = \bigl\{\mathbf{j} + L\mathbf{z}\colon\ \mathbf{z} \in \mathbb{Z}^d\bigr\}.$$

Note that the $J(L, \mathbf{j})$ are disjoint lattices for $\mathbf{j} \in I$ and $\bigcup_{\mathbf{j} \in I} J(L, \mathbf{j}) = \mathbb{Z}^d$.

Corollary 12. *Let $S_{\mathbf{j}} = \sum\{w_l\colon\ l \in \{1, \dots, n\},\ \mathbf{c}_l \in J(L, \mathbf{j})\}$. If $S_{\mathbf{j}} = L^{-d}$ for all $\mathbf{j} \in I$ then $\mu \ll \mathcal{L}$.*

We prove this corollary only for $d = 1$. In this case we have $J(L, t) = \{t + Lz\colon\ z \in \mathbb{Z}\}$ for $t = 0, \dots, L-1$. The proof for $d > 1$ is similar by considering an expression for the indices with d coordinates.

Proof. We show that $S_t = L^{-1}$ for all t if and only if $\mathbf{e}Z_m = L^{-1}\mathbf{e}$ for all m, and the result follows from Corollary 11.

Let $j \in \{1, \dots, N\}$. We have $\langle j\rangle = I(j_0) = [j_0 - 1, j_0]$ for some integer j_0, and without loss of generality we assume $j_0 > 0$. If $c_l \in J(L, t)$ then

$$c_l = t + Lz$$

for some $z \in \mathbb{Z}$. From (10) we have $Z_m(i, j) = w_l$ for

$$m = j_0 - t - L \cdot \operatorname{floor}\bigl((j_0 - 1 - t)/L\bigr)$$

and some $i \in \{1, \dots, N\}$ irrelevant here. Note that m depends only on j_0 and t, but does not depend on z and hence it does not depend on the specific l. Since $L \cdot \operatorname{floor}(y/L) = y - y \pmod L$ for y integer, we have $m = m(j_0, t) = (j_0 - 1 - t) \pmod L + 1$. It is trivial to check that $\{m(j_0, t)\colon\ t = 0, \dots, L-1\} = \{1, 2, \dots, L\} = M$.

From this it follows that, for all j, the sum of the jth column of Z_m for each $m \in M$ is equal to S_t for some t, and there follows the claimed equivalence. $\square$

We conclude this section with a result about the Lebesgue measure of the self-similar compact set E. If $\mu \ll \mathcal{L}$ then $\mathcal{L}E > 0$, since $\mu E = 1 > 0$. We show that actually $\mathcal{L}E \geqslant 1$.

Corollary 13. *If $\mu \ll \mathcal{L}$ then $\mathcal{L}E \geqslant 1$.*

Proof. If $\mu \ll \mathcal{L}$ then η is the Lebesgue measure on $\langle 0\rangle = [0, 1]^d$, by Proposition 10. Thus $\eta \ll \mathcal{L}$, and its density function is $I_{\langle 0\rangle}$, with $I_{\langle 0\rangle}(x) = 1$ if $x \in \langle 0\rangle$ and 0 otherwise. Let f be the density function of μ, with $\mu D = \int_D f\,d\mathcal{L}$. Since $\eta = \sum_{j=1}^N \mu_j \circ g_j$ we have

$$\sum_{j=1}^N f \circ g_j = I_{\langle 0\rangle}$$

in $\langle 0\rangle$ (almost surely). Let $A = \{x \in E\colon\ f(x) > 0\}$. Since $I_{\langle 0\rangle}$ is positive in $\langle 0\rangle$ and $\mathcal{L}\langle 0\rangle = 1$ it must be $\mathcal{L}A \geqslant 1$, and the result follows since $A \subset E$. $\square$

Kenyon [14, Lemma 5] obtains $\mathcal{L}E = 1$ in the case with $d = 1$, $n = 3$, $c_1 = 0$, $c_2 = p$, $c_3 = q$, with $(p + q) \pmod 3 = 0$ and p, q being coprime, and $L = 3$. With Corollary 12, it is easy to check that in this case $\mu \ll \mathcal{L}$ for $w_1 = w_2 = w_3 = 3^{-1}$. Hence, in this case, equality is achieved in the inequality $\mathcal{L}E \geqslant 1$ of Corollary 13.

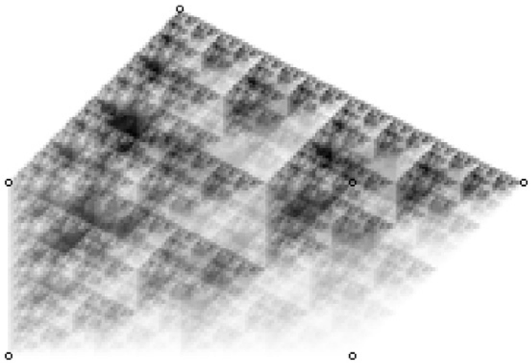

Fig. 1. The centres are $(0,0)$, $(0,1)$, $(1,2)$, $(2,1)$, $(3,1)$ and $(2,0)$, $L=2$ and $k=6$. The weights w_i are $\frac{2}{12}$, $\frac{4}{20}$, $\frac{1}{4}$, $\frac{1}{20}$, $\frac{1}{4}$, $\frac{1}{12}$, respectively, and from Corollary 12 it follows that μ is absolutely continuous.

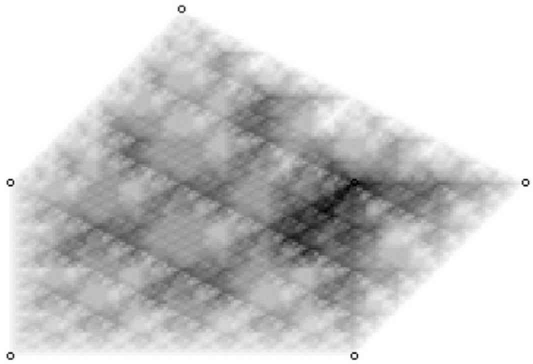

Fig. 2. The centres are $(0,0)$, $(0,1)$, $(1,2)$, $(2,1)$, $(3,1)$ and $(2,0)$, $L=2$ and $k=6$. The weights w_i are $\frac{1}{6}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{1}{6}$ and $\dim\mu \simeq 1'997$, approached with three decimal digits of precision in the usual way. This is obtained from Theorem 9 and the explanations given in Section 4.

4. Examples

Since V is a stationary hidden Markov chain, we can obtain lower and upper bounds for the entropy in Theorem 9 from the converging sequences in (3) and (4) (see Section 1).

To calculate the terms of the non-increasing sequence we use:

$$H(V_1,\dots,V_k) = -\sum_{\mathbf{i}\in M^k} \big(\mathbf{e}Z_{\mathbf{i}}\mathbf{p}^t\big)\log_2\big(\mathbf{e}Z_{\mathbf{i}}\mathbf{p}^t\big).$$

To obtain a non-decreasing sequence requires the consideration of a Markov chain such that V is a function of it. It can be shown that there is a Markov chain X such that V is a function of it and with

$$H(X_1,V_2,\dots,V_k) = -\sum_{i_1=1}^{L^d N}\sum_{\mathbf{i}\in M^{k-1}} \big(\mathbf{e}Z^*_{i_1}Z_{\mathbf{i}}\mathbf{p}^t\big)\log_2\big(\mathbf{e}Z^*_{i_1}Z_{\mathbf{i}}\mathbf{p}^t\big),$$

where $\{Z^*_{i_1}\colon\ i_1=1,\dots,N\cdot L^d\} = \{Z_iA_j\colon\ i=1,\dots,L^d,\ j=1,\dots,N\}$ and A_j is the $N\times N$ matrix with 1 in the (j,j) entry and zero elsewhere. We omit the details, as they are similar to those in [23, Remark 4], which are for a particular case.

In some cases we obtain many digits of precision for $\dim\mu$ with a low computation time, as in [23], but in other cases the gaps between the terms of both sequences are not so small for moderate k.

In some of the cases considered in [15,18] the absolute continuity of the self-similar measure can be easily obtained from Corollary 12.

In [18] there are considered the cases where $d=1$ and $L=2$, and for these cases absolute continuity is shown when n is even, $c_l=l$ and $w_l=n^{-1}$, $1\leqslant l\leqslant n$ (Corollary 3.4). We obtain absolute continuity if

$$\sum\{w_l\colon\ c_l \text{ is even}\} = \sum\{w_l\colon\ c_l \text{ is odd}\} = \frac{1}{2},$$

for all n, which improves on the result of [18].

In [15] there are considered the cases where $d=1$, $n=L+1$ and $c_l=l$, $1\leqslant l\leqslant n$. We obtain absolute continuity if $w_1+w_n=w_2=\dots=w_{n-1}=L^{-1}$.

We conclude this section with graphical representations for some homogeneous rational self-similar measures in $\mathbb{R}^2$.

For the indicated values of k we calculate $\mu\langle j;\mathbf{i}\rangle = \mathbf{e}_jZ_{\mathbf{i}}\mathbf{p}^t$ for $j\in\{1,\dots,N\}$ and $\mathbf{i}\in M^k$. We then colour the square $\langle j;\mathbf{i}\rangle$ in a grey tone with degree of darkness ranging from white, when the measure is null, to black, when $\mu\langle j;\mathbf{i}\rangle$ takes the maximum value among all the $\mu\langle j';\mathbf{i}'\rangle$ calculated. The centres of the small circles in the picture are the centres of the homotheties (see Figs. 1–4).

5. Conclusion

We have studied some self-similar measures with overlaps. We have obtained the dimension of such a measure as a Shannon entropy (up to constants), which is a new instance of a result linking dimension and entropy. This approach allows us to base the determination of the absolute continuity or singularity of the measures on the fact that the discrete uniform distribution maximizes entropy. In addition, the approach allows us to obtain lower and upper bounds for the dimension.

We have not studied the multifractal properties, only the dimension. We have improved previous results by other authors for some of the cases studied here.

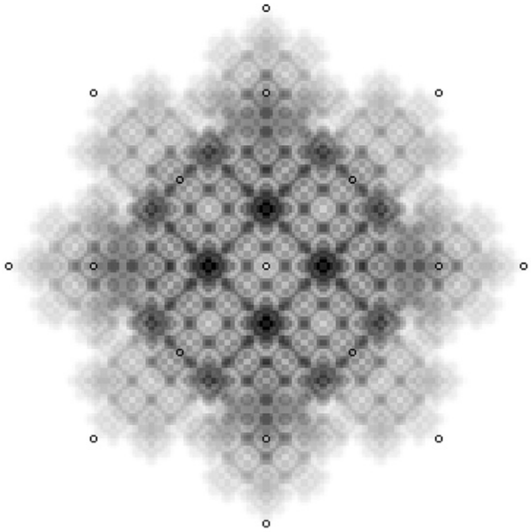

Fig. 3. The centres are $(3,3)$, $(2,2)$, $(4,4)$, $(2,4)$, $(4,2)$, $(3,0)$, $(5,1)$, $(6,3)$, $(3,6)$, $(5,5)$, $(1,5)$, $(0,3)$, $(1,1)$, $(3,5)$, $(3,1)$, $(1,3)$ and $(5,3)$, $L=3$, $k=3$. The weights w_i are $\frac{1}{18}$, $\frac{1}{16}$, $\frac{1}{16}$, $\frac{1}{16}$, $\frac{1}{16}$, $\frac{1}{18}$, $\frac{1}{18}$, $\frac{1}{18}$, $\frac{1}{18}$, $\frac{1}{18}$, $\frac{1}{18}$, $\frac{1}{18}$, $\frac{1}{18}$, $\frac{1}{16}$, $\frac{1}{16}$, $\frac{1}{16}$, $\frac{1}{16}$. We have $1'9660 < \dim\mu < 1'9989$.

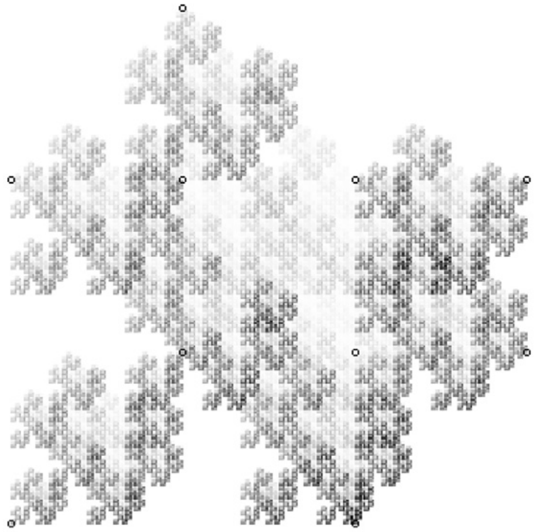

Fig. 4. The centres are $(0,2)$, $(1,3)$, $(1,2)$, $(3,2)$, $(2,1)$, $(3,1)$, $(1,1)$, $(2,2)$, $(0,0)$ and $(2,0)$, $L=3$, $k=4$. The weights w_i are $\frac{3}{30}$, $\frac{3}{30}$, $\frac{1}{30}$, $\frac{4}{30}$, $\frac{1}{30}$, $\frac{4}{30}$, $\frac{4}{30}$, $\frac{1}{30}$, $\frac{4}{30}$, $\frac{5}{30}$. We have $1'87 < \dim\mu < 1'97$.

Work in progress seems to indicate that these ideas can be extended to a wider range of cases.

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